MATH-GA2450 Complex Analysis

Compact Sets
Limit of Function
Continuity and Uniform Continuity

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September 24, 2024

Compact Sets

- A subset $S \subset \mathbb{C}$ is **compact** if for any sequence $(z_n : \ n \in \mathbb{Z}_+) \subset S$, there exists a subsequence $(z_{n_k} : \ k \in \mathbb{Z}_+)$ such that is convergent and

$$\lim_{k \to \infty} z_{n_k} \in S$$

- A subset $S \subset \mathbb{C}$ is compact if and only if it is closed and bounded
- A subset $S \subset \mathbb{C}$ is compact if and only if the following holds: Given any countable collection of open subsets $O_1, O_2, \ldots$ of $\mathbb{C}$, such that

$$S \subset O_1 \cup O_2 \cup \cdots ,$$

there exists a finite subcollection $O_{n_1}, \ldots, O_{n_m}$ such that

$$S \subset O_{n_1} \cup O_{n_2} \cup \cdots \cup O_{n_k}$$

Limit of a Function

- Let $f : S \to \mathbb{C}$ be a function
- Given $w \in \overline{S}$ and $L \in \mathbb{C}$, we say

$$\lim_{z \to w} f(z) = L$$

if for any sequence $(z_n : \ n \in \mathbb{Z}_+) \subset S$ such that

$$\lim_{n \to \infty} z_n = w,$$

the following holds:

$$\lim_{n \to \infty} f(z_n) = L$$

Continuous Functions

- Let $S \subset \mathbb{C}$
- A function $f : S \to \mathbb{C}$ is **continuous** if for each $w \in S$,

$$\lim_{z \to w} f(z) = f(w)$$

- A function $f : S \to \mathbb{C}$ is continuous if and only if for any open subset $O \subset \mathbb{C}$, $f^{-1}(O) \subset S$ is relatively open, i.e., there exists an open subset $U \subset \mathbb{C}$ such that

$$f^{-1}(O) = U \cap S$$

- If $f : S \to \mathbb{C}$ is continuous, then for any compact subset $C \subset S$, $f(C) \subset \mathbb{C}$ is compact
- If $f : S \to \mathbb{R}$ is continuous, then for any compact subset $C \subset S$, there exists $z_{\text{min}}, z_{\text{max}} \in C$ such that for any $z \in C$,

$$f(z_{\text{min}}) \leq f(z) \leq f(z_{\text{max}})$$

Continuity and Uniform Continuity

- A function $f : S \to \mathbb{C}$ is continuous if and only if the following holds: For each $z \in S$ and for any $\epsilon > 0$, there exists $\delta(z, \epsilon) > 0$ such that

$$|w - z| \leq \delta(z, \epsilon) \implies |f(w) - f(z)| \leq \epsilon$$

- A function $f : S \to \mathbb{C}$ is **uniformly continuous** if and only if the following holds: For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $z \in S$,

$$|w - z| \leq \delta(\epsilon) \implies |f(w) - f(z)| \leq \epsilon$$

Examples

- The function

$$\begin{aligned} D(0,2) &\to \mathbb{C} \\ z &\mapsto z^2 \end{aligned}$$

  is uniformly continuous
- The function

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^2 \end{aligned}$$

  is not uniformly continuous
- The function

$$\begin{aligned} D(0,1)\setminus\{0\} &\to \mathbb{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

  is not uniformly continuous

On Compact Domain, Continuity $\implies$ Uniform Continuity (Part 1)

- Theorem: A continuous function $f : C \to \mathbb{C}$, where $C \subset \mathbb{C}$ is compact, is uniformly continuous
- Proof:
  - Fix any $\epsilon > 0$
  - For any $z \in \mathbb{C}$, there exists $\delta_z > 0$ such that

$$\forall w \in C, \ |w - z| < \delta_z \implies |f(w) - f(z)| < \frac{\epsilon}{2}$$

  - Therefore, for any $w_1, w_2 \in D(z, \delta_z)$,

$$|f(w_2) - f(w_1)| \leq |f(w_2) - f(z)| + |f(w_1) - f(z)| < \epsilon$$

On Compact Domain, Continuity $\implies$ Uniform Continuity (Part 2)

- On other hand,
$$C \subset \bigcup_{z \in C} D(z, \delta_z)$$
- C compact implies there exists $z_1, \ldots, z_N \in C$ such that
$$C \subset D(z_1, \delta_1) \cup \cdots \cup D(z_N, \delta_N), \text{ where } \delta_k = \delta_{z_k}$$
- By assumption, if $w_1, w_2 \in D(z_k, \delta_k)$, then
$$|f(w_2) - f(w_1)| < \epsilon$$
- Let
$$\delta = \frac{1}{2}\min(\delta_1, \ldots, \delta_N)$$

On Compact Domain, Continuity $\implies$ Uniform Continuity (Part 3)

- For any $\epsilon > 0$, let δ be as given above
- For any $w_1 \in C$, there exists $1 \leq k \leq N$ such that $w_1 \in D(z_k, \delta_k)$
- If $|w_2 - w_1| < \delta$, then

$$|w_2 - z_k| \leq |w_2 - w_1| + |w_1 - z_k| < 2\delta < \delta_k$$

- It follows that $w_1, w_2 \in D(z_k, \delta_k)$, and therefore,

$$|f(w_2) - f(w_1)| < \epsilon$$

- This shows that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $w_1, w_2 \in C$, then

$$|w_2 - w_1| < \delta \implies |f(w_2) - f(w_1)| < \epsilon$$