MATH-GA2450 Complex Analysis

Ratio of Convergence for Product of Power Series
Analytic Functions
Uniqueness of Power Series

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Product of Power Series (Part 1)

- Consider two power series

$$\sum_{k=0}^{\infty} a_k z^k \text{ with radius of convergence } R > 0$$

$$\sum_{k=0}^{\infty} b_k z^k \text{ with radius of convergence } S > 0$$

- If $|z| < \min(R, S)$, then the power series converge absolutely
- I.e.,

$$A = \sum_{k=0}^{\infty} |a_k z^k| \text{ and } B = \sum_{k=0}^{\infty} |b_k z^k|$$

  converge
- We want to show that if $|z| < \min(R, S)$, then the series

$$\sum_{k=0}^{\infty} c_k z^k \text{ where } c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

  converges absolutely

Product of Power Series (Part 2)

- (math.stackexchange.com) Assume $|z| < \min(R, S)$ and let

$$
\begin{aligned}
C_N &= \sum_{k=0}^{N} |c_k z^k| \\
&= \sum_{k=0}^{N} \left| \sum_{j=0}^{k} a_j b_{k-j} z^k \right| \\
&\leq \sum_{k=0}^{N} \left(\sum_{j=0}^{k} (|a_j z^j||b_{k-j} z^{k-j}| \right) \\
&= \sum_{j=0}^{N} |a_j||z|^j \sum_{i=0}^{N-j} |b_i||z|^i \\
&\leq \sum_{j=0}^{N} |a_j||z|^j B \\
&\leq AB
\end{aligned}
$$

Product of Power Series (Part 3)

- The sequence $(C_N : \; N \geq 0)$ is increasing and bounded
- Lemma: A bounded increasing sequence of reals converges
- It follows that if $|z| < \min(R, S)$, then

$$\sum_{k=0} |c_k z^k| \text{ converges,}$$

- Therefore, the radius of convergence for

$$\sum_{k=0}^{\infty} c_k z^k$$

is at least $\min(R, S)$

Complex Analytic Functions

- Let $O \subset \mathbb{C}$ be open
- A function $f : O \to \mathbb{C}$ is **(complex) analytic** if for each $z_0 \in O$, there exists $r > 0$ and a power series

$$\sum_{k=0} a_k(z - z_0)^k$$

such that for any $z \in D(z_0, r)$, the power series converges absolutely and

$$f(z) = \sum_{k=0} a_k(z - z_0)^k$$

Uniqueness of Power Series

- Given an analytic function $f : O \to \mathbb{C}$ and $z_0 \in O$, suppose

$$f(x) = \sum_{k=0}^{\infty} a_k (z -_0)^k = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

- Is it true that

$$\forall k \geq 0,\ a_k = b_k?$$

- Equivalently, if we set $c_k = b_k - a_k$, does there exist a power series

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k$$

that is equal to zero on an open set containing 0 but has at least one nonzero coefficient c_k?

Power Series that is Nonzero at Origin (Part 1)

- Consider a power series with positive radius of convergence R,

$$\sum_{k=0}^{\infty} c_k z^k \text{ such that } c_0 \neq 0,$$

- Let $0 < s < r < R$
- The series

$$M = \sum_{k=1}^{\infty} |c_k| r^{k-1} = \frac{1}{r} \left(\sum_{k=0}^{\infty} |c_k| r^k - |c_0| \right)$$

converges

Power Series that is Nonzero at Origin (Part 2)

- If $|z| < s$, then

$$\left|\sum_{k=0}^{\infty} c_k z^k\right| = \left|c_0 + z\sum_{k=1}^{\infty} c_k z^{k-1}\right| \geq |c_0| - \left|z\sum_{k=1}^{\infty} c_k z^{k-1}\right|$$
$$\geq |c_0| - |z|\sum_{k=1}^{\infty} |c_k||z|^{k-1} \geq |c_0| - s\sum_{k=1}^{\infty} |c_k|r^{k-1}$$
$$\geq |c_0| - Ms$$

- If $s < \frac{|c_0|}{2M}$, then for all $z \in D(0,s)$,

$$\left|\sum_{0}^{\infty} c_k z^k\right| > \frac{|c_0|}{2} > 0$$

- Therefore, if $z \in D(0,s)$, then $\sum_{k=0}^{\infty} c_k z^k \neq 0$

Nonzero Power Series that Vanishes at Origin

- Consider a power series

$$\sum_{k=k_0}^{\infty} c_k z^k = z^{k_0} \sum_{k=k_0} c_k z^{k-k_0} = z^{k_0} \sum_{j=0}^{\infty} c_{k_0+j} z^j,$$

  where $c_{k_0} \neq 0$
- We have shown that there exists $s > 0$ such that if $|z| < s$, then

$$\sum_{j=0}^{\infty} c_{k_0+j} z^j \neq 0$$

- Therefore, if $z \in D(0,s) \backslash \{0\}$, then

$$\sum_{k=k_0}^{\infty} c_k z^k \neq 0$$

Contrapositive

- If there exists a sequence $(z_j : j \geq 1)$ such that

$$\lim_{j \to \infty} z_k = 0 \text{ and } \sum_{k=0}^{\infty} c_k z_j^k = 0,$$

  then for all $k \geq 0$, $c_k = 0$
- **Corollary:** If there exists an open $O \subset \mathbb{C}$ containing z_0 and a power series

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k$$

  that vanishes for all $z \in O$, then $c_k = 0$ for all $k \geq 0$
- **Corollary:** If there exists an open $O \subset \mathbb{C}$ containing z_0 and two power series such that

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} c_k (z - z_0)^k \text{ for all } z \in O,$$

  then $a_k = b_k$ for all $k \geq 0$