MATH-GA2450 Complex Analysis

Contour Integral of Laurent Series

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Norm of Integral $\leq$ Integral of Norm

- If $f : [a, b] \to \mathbb{C}$ is continuous, then

$$\left| \int_{t=a}^{t=b} f(t)\, dt \right| \leq \int_{t=a}^{t=b} |f(t)|\, dt$$

- If $|f| \leq |g|$, then

$$\int_{t=a}^{t=b} |f(t)|\, dt \leq \int_{t=a}^{t=b} |g(t)|\, dt$$

- Both are easily proved using definition of Riemann integral in terms of upper and lower Riemann sums and triangle inequality

Upper Bound for Integral of Continuous Function (Part 1)

- Let $O \subset \mathbb{C}$ be open, $f : O \to \mathbb{C}$ be continuous, and $c : [a, b] \to O$ be a piecewise C^1 curve
- Since $[a, b]$ is compact, so is $c([a, b])$
- Therefore, $f(c([a, b]))$ is bounded
- Denote the **sup norm** of f on the curve c to be

$$\|f\|_c = \sup\{|f(t)| : \ t \in [a, b]\}$$

- The length of c is defined to be

$$L(c) = \int_{t=a}^{t=b} |c'(t)| \, dt$$

Upper Bound for Integral of Continuous Function (Part 2)

- Then

$$
\begin{aligned}
\left| \int_c f(z)\, dz \right| &= \left| \int_{t=a}^{t=b} f(c(t)) c'(t)\, dt \right| \\
&\leq \int_{t=a}^{t=b} |f(c(t)) c'(t)|\, dt \\
&= \int_{t=a}^{t=b} |f(c(t))| |c'(t)|\, dt \\
&\leq \int_{t=a}^{t=b} \|f\|_c |c'(t)|\, dt \\
&= \|f\|_c \int_{t=a}^{t=b} |c'(t)|\, dt \\
&= \|f\|_c L(c)
\end{aligned}
$$

Uniform Convergence of Sequence of Functions

- Let $K \subset \mathbb{C}$ be compact
- The set $f(K) \subset \mathbb{C}$ is compact and therefore

$$\|f\|_K = \sup\{|f(z) : \; z \in K\}$$

  is finite
- A sequence of continuous functions $f_k : K \to \mathbb{C}$ converges uniformly to $f : K \to \mathbb{C}$ if and only if

$$\lim_{k \to \infty} \|f_k - f\|_K = 0$$

- The limit f is continuous

Uniform Convergence of Sequence of Functions

- Let $O \subset \mathbb{C}$ be open and $c : [a, b] \to O$ be a continuous curve
- Since $[a, b]$ is compact, so is $c([a, b])$
- Let $f_k : O \to \mathbb{C}$ be a sequence of continuous functions that converge uniformly to $f : O \to \mathbb{C}$
- In other words, for any $\epsilon > 0$, there exists N such that

$$k > N \implies \forall z \in O,\ |f_k(z) - f(z)| < \epsilon$$

- If we define

$$\|f\|_\infty = \sup\{|f(z)| \ :\ z \in O\},$$

then

$$f_k \to f \text{ uniformly}$$

if and only if

$$\lim_{k\to\infty} \|f_k - f\|_\infty = 0$$

Integral of Uniformly Convergent Sequence of Functions

- Let $c : [a, b] \to O$ be a piecewise C^1 curve
- Let $f_k : O \to \mathbb{C}$ be continuous functions that converge uniformly to a function $f : O \to \mathbb{C}$
- Then

$$\begin{aligned}\left|\int_c f_k(z)\,dz - \int_c f(z)\,dz\right| &= \left|\int_c f_k(z) - f(z)\,dz\right| \\ &\le \|f_k(z) - f(z)\|_K L(c)\end{aligned}$$

and therefore,

$$\begin{aligned}\lim_{k\to\infty}\left|\int_c f_k(z)\,dz - \int_c f(z)\,dz\right| &\le L(c)\lim_{k\to\infty}\|f_k(z) - f(z)\|_K \\ &= 0\end{aligned}$$

- It follows that

$$\lim_{k\to\infty}\int_c f_k(z)\,dz = \int_c f(z)\,dz$$

Integral of Uniformly Convergent Series of Functions

- Then, given a piecewise C^1 curve $c : [a, b] \to O$,

$$\begin{aligned}\int_c \sum_{k=1}^{\infty} f_k(z)\,dz &= \int_c \lim_{N\to\infty} F_N(z)\,dz \\ &= \lim_{N\to\infty} \int_c F_N(z)\,dz) \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \int_c f_k(z)\,dz) \\ &= \sum_{k=1}^{\infty} \int_c f_k(z)\,dz\end{aligned}$$

Uniformly Convergent Series of Functions

- For each $k \in \mathbb{Z}_+$, let $f_k : O \to \mathbb{C}$ be a continuous function
- For each $N \in \mathbb{Z}_=$, let $F_N : O \to \mathbb{C}$ be the continuous function given by

$$F_N(z) = \sum_{k=1}^{N} f_k(z)$$

- The series

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly to $F : O \to \mathbb{C}$ if

$$F_N \to F \text{ uniformly}$$

Partial Sums of Laurent Series Converge Uniformly (Part 1)

- Consider a Laurent series

$$f(z) = \sum_{k=k_0}^{\infty} a_k (z - z_0)^k$$

that converges absolutely for each $z \in D(z_0, R)\backslash\{0\}$

- By setting $z = z_0 + r$, where $0 < r < R$, it follows that

$$\sum_{k=k_0}^{\infty} |a_k| r^k \text{ converges}$$

Partial Sums of Laurent Series Converge Uniformly (Part 2)

- It follows that if $z \in D(z_0, r)$,

$$|f(z) - S_N(z)| = \left| \sum_{k=N+1}^{\infty} a_k (z - z_0)^k \right|$$
$$\leq \sum_{k=N+1}^{\infty} |a_k||z - z_0|^k$$
$$\leq \sum_{k=N+1}^{\infty} |a_k| r^k$$
$$= \sum_{k=k_0}^{\infty} |a_k| r^k - \sum_{k=k_0}^{N} |a_k| r^k$$

and therefore

$$\lim_{N\to\infty} \|f - S_N\|_\infty \leq \lim_{N\to\infty} \sum_{k=k_0}^{\infty} |a_k| r^k - \sum_{k=k_0}^{N} |a_k| r^k = 0$$

Contour Integral of Laurent Series (Part 1)

- It follows that if $c : [a, b] \to D(z_0, r)$ is piecewise C^1 ,

$$\begin{aligned}
\int_c f(z)\,dz &= \int_c \sum_{k=k_0}^{\infty} a_k (z - z_0)^k\,dz \\
&= \int_c \lim_{N\to\infty} S_N(z)\,dz \\
&= \lim_{N\to\infty} \int_c S_N(z)\,dz \\
&= \lim_{N\to\infty} \int_c \sum_{k=k_0}^{N} a_k (z - z_0)^k\,dz \\
&= \lim_{N\to\infty} \sum_{k=k_0}^{N} \int_c a_k (z - z_0)^k\,dz \\
&= \sum_{k=k_0}^{\infty} a_k \int_c (z - z_0)^k\,dz
\end{aligned}$$

Contour Integral of Laurent Series (Part 2)

- Consider a Lauren series $\sum_{k=-n} a_k(z-z_0)^k$
- If $k \neq -1$, then the function $(z-z_0)^k$ has an antiderivative
- It follows that if $c : [a, b] \to D(z_0, r$ is a closed curve,

$$\begin{aligned}
\int_c \sum_{k=-n}^{\infty} a_k z^k \, dz &= \int_c \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \, dz \\
&= \sum_{k=-n}^{-1} \int_c a_k z^k \, dx + \sum_{k=0}^{\infty} \int_c a_k z^k \\
&= 2\pi i \int_c \frac{a_{-1}}{z} \, dz
\end{aligned}$$