MATH-GA2450 Complex Analysis

Homotopic Curves
Homotopy Forms of Cauchy's Theorem

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October 17 2024

Fundamental Theorem of Calculus for Holomorphic Functions

- Let $O \subset \mathbb{C}$ and $c : [a, b] \to O$ be a piecewise C^1 curve
- If $F : O \to \mathbb{C}$ is holomorphic, then

$$\int_c F'(z)\,dz = F(c(b)) - F(c(a))$$

- If $f : O \to \mathbb{C}$ is holomorphic and has an antiderivative F, then

$$\int_c f(z)\,dz = F(c(b)) - F(c(a))$$

- **Proof.**

$$\begin{aligned}\int_c f(z)\,dz &= \int_{t=a}^{t=b} f(c(t))c'(t)\,dt \\ &= \int_{t=a}^{t=b} \frac{d}{dt} f(c(t))\,dt \\ &= f(c(b)) - f(c(a))\end{aligned}$$

- Crucial ingredient: Chain rule for holomorphic functions

Corollaries

- Let $f : O \to \mathbb{C}$ have an antiderivative $F : O \to \mathbb{C}$
- Given piecewise C^1 curves

$$c_1 : [a_1, b_1] \to O \text{ and } c_2 : [a_2, b_2] \to O$$

  such that

$$c_1(a_1) = c_2(a_2) \text{ and } c_1(b_1) = c_2(b_2),$$

  the following holds:

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

- If $c : [a, b] \to O$ is closed, then

$$\int_c f(z)\,dz = 0$$

Homotopy of Two Curves

- Let $O \subset \mathbb{C}$ be open
- Given continuous curves $c_0 : [a, b] \to O$ and $c_1 : [a, b] \to O$, a **homotopy** is a continuous map

$$C : [0, 1] \times [a, b] \to O$$

such that

$$\forall t \in [a, b], C(0, t) = c_0(t) \text{ and } C(1, t) = c_1(t)$$

- If c_0 and c_1 have the same endpoints, i.e.,

$$c_0(a) = c_1(a) \text{ and } c_0(b) = c_1(b),$$

then they are **homotopic** if the exists a homotopy such that

$$\forall s \in [0, 1], C(s, a) = c_0(a) \text{ and } C(s, b) = c_0(b)$$

- Two continuous closed curves c_0 and c_1 are **homotopic** if

$$\forall s \in [0, 1], C(s, a) = C(s, b)$$

- Whether two curves are homotopic or not depends on both the curves and the codomain O

Smooth Homotopy of Continuous Curves

- We will assume the following without proof
- If $c_0, c_1 : [a, b] \to O$ are homotopic, then there exists a homotopy
$$C : [0, 1] \times [a, b] \to O$$
such that the following hold:
  - C restricted to $(0, 1) \times (a, b)$ is C^2
  - There exists a constant $M > 0$ such that for any $(s, t) \in [0, 1] \times [a, b]$,
$$|\partial^2_{st} C(s, t)| \leq M$$
- Below, we will always assume that a homotopy C satisfies these conditions

Examples

- A curve is homotopic to itself
- Any two curves in $\mathbb{C}$ are homotopic
- If O is connected and $z_0, z_1 \in O$, then the curves $c_0, c_1 : [a, b] \to O$ given by

$$c_0(t) = z_0 \text{ and } c_1(t) = z_1$$

  are homotopic
- Consider the curve $c_0 : [0, 2\pi] \to \mathbb{C}$ given by

$$c_0(t) = e^{it}$$

  - c_0 is homotopic to the constant curve $c_1 : [0, 2\pi] \to \mathbb{C}$ given by $c_1(t) = 0$
  - On the other hand, we will prove that the curve c_0 in $O = \mathbb{C}\backslash\{0\}$ is **not** homotopic to c_1

Homotopy Forms of Cauchy's Theorem

- **Theorem 1.** If $O \subset \mathbb{C}$ is open, $f : O \to \mathbb{C}$ is holomorphic, and c_0, c_1 are homotopic piecewise C^1 curves with the same endpoints , then

$$\int_{c_0} f(z)\,dz = \int_{c_1} f(z)\,dz$$

Cauchy's Theorem for Homotopic Closed Curves

- **Theorem 2.** If $O \subset \mathbb{C}$ is open, $f : O \to \mathbb{C}$ is holomorphic, and

$$c_0, c_1 : [a, b] \to O$$

are homotopic closed curves, then

$$\int_{c_0} f(z)\, dz = \int_{c_1} f(z)\, dz$$

Proof of Theorems 1 and 2 (Part 1)

- Let $O \subset \mathbb{C}$ be open and $f : O \to \mathbb{C}$ be holomorphic
- Let $c_0, c_1 : [a, b] \to O$ be homotopic curves and $C : [0, 1] \times [a, b] \to O$ be a homotopy between them
- For each $s \in [0, 1]$, let

$$I(s) = \int_{C(s,\cdot)} f(z)\, dz = \int_{t=a}^{t=b} f(C(s,t))\partial_t C(s,t)\, dt$$

- It is straightforward to show that $I : [0, 1] \to \mathbb{C}$ is continuous
- Also,

$$I(0) = \int_{c_0} f(z)\, dz \text{ and } I(1) = \int_{c_1} f(z)\, dz$$

- To prove the theorems, it suffices to prove that for each $s \in (0, 1)$, $I'(s) = 0$

Proof of Theorems 1 and 2 (Part 2)

- The assumptions on C imply that in the calculation below, the differentiation and integration can be interchanged:

$$\begin{aligned} I'(s) &= \partial_s \int_{t=a}^{t=b} f(C(s,t))\partial_t C(s,t)\,dt \\ &= \int_{t=a}^{t=b} \partial_s(f(C(s,t))\partial_t C(s,t))\,dt \end{aligned}$$

- Key calculation

$$\begin{aligned} &\partial_s(f(C(s,t))\partial_t C(s,t)) \\ &= f'(C(s,t))\partial_s C(s,t)\partial_t C(s,t)) + f(C(s,t))\partial_s(\partial_t C(s,t) \\ &= f'(C(s,t))\partial_s C(s,t)\partial_t C(s,t)) + f(C(s,t))\partial_t(\partial_s C(s,t) \\ &= \partial_t(f(C(s,t))\partial_s C(s,t)) \end{aligned}$$

Proof of Theorem 1

- Assume that for each $s \in [0, 1]$,

$$C(s, a) = c_0(a) \text{ and } C(s, b) = c_0(b)$$

- It follows that

$$\partial_s C(s, a) = \partial_s C(s, b) = 0$$

  and therefore $I'(s) = 0$
- This implies that

$$\int_{c_0} f(z)\, dz = I(0) = I(1) = \int_{c_1} f(z)\, dz,$$

  which proves Theorem 1

Proof of Theorem 2

- Since c_0 and c_1 are closed, there exists, by assumption, a homotopy C between c_0 and c_1 such that for each $s \in [0, 1]$,

$$C(s, a) = C(s, b) \text{ and } \partial_s C(s, a) = \partial_s C(s, b)$$

- This implies that

$$\int_{c_0} f(z)\, dz = I(0) = I(1) = \int_{c_1} f(z)\, dz,$$

which proves Theorem 2

Explanation of Key Calculation

- If f has an antiderivative F, then

$$\begin{aligned}\partial_s(f(C(s,t))\partial_t C(s,t)) &= \partial_s(\partial_t F(C(s,t))) \\ &= \partial_t(\partial_s F(C(s,t))) \\ &= \partial_t(f(C(s,t))\partial_s C(s,t))\end{aligned}$$

Cauchy's Theorem for Null-Homotopic Curve

- **Definition.** A closed curve $c : [0,1] \to O$ is **null-homotopic** if it is homotopic to a point $z_0 \in O$, i.e., there exists a homotopy $C : [0,1] \times [a,b] \to O$ such that for each $t \in [a,b]$,

$$C(0,t) = c(t) \text{ and } C(1,t) = z_0$$

- **Corollary.** If $O \subset \mathbb{C}$ is open and $c : [a,b] \to O$ is null-homotopic, then for any holomorphic $f : O \to \mathbb{C}$,

$$\int_c f(z)\, dz = 0$$

- This follows directly by Theorem 2

Cauchy's Theorem For Simply Connected Domain

- An open $O \subset \mathbb{C}$ is **simply connected** if it is connecccted and any two curves with the same endpoints are homotopic
- Equivalently, O is **simply connected** if it is connected, and any closed curve in O is null-homotopic
- **Theorem 3.** If $O \subset \mathbb{C}$ is simply connected and open, $f : O \to \mathbb{C}$ is holomorphic, and $c_0, c_1 : [0,1] \to \mathbb{C}$ are piecewise C^1 curves that are homotopic with fixed endpoints, then
$$\int_{c_0} f(z)\, dz = \int_{c_1} f(z)\, dz$$
or, equivalently, if $c : [0,1] \to O$ is a closed curve, then
$$\int_{c} f(z)\, dz = 0$$

Antiderivative of Holomorphic Function on Simply Connected Domain

- **Corollary.** Any holomorphic function f on a simply connected domain O has an antiderivative $F : O \to \mathbb{C}$
- **Proof.** Let $z_0 \in \mathbb{C}$ and define $F : O \to \mathbb{C}$ by

$$F(z) = \int_c f(z)\, dz,$$

where $c : [0,1] \to O$ is a piecewise C^1 curve such that $c(0) = z_0$ and $c(1) = z$

Logarithm Function Does Not Exist on $\mathbb{C}\backslash\{0\}$

- **Definition.** A logarithm function is an antiderivative of the function
$$f(z) = \frac{1}{z}$$
- There is no logarithm function on $\mathbb{C}\backslash\{0\}$, because if c is the unit circle, then
$$\int_c \frac{dz}{z} = 2\pi i \neq 0$$
- $\mathbb{C}\backslash\{0\}$ is not simply connected

Branch Cut

- Let
$$\mathbb{C}\backslash[0,\infty) = \{x+iy : \; y \neq 0 \text{ or } x < 0\}$$
- $\mathbb{C}\backslash[0,\infty)$ is simply connected
- It follows that there exists a holomorphic function
$$\log : \mathbb{C}\backslash[0,\infty) \to \mathbb{C}$$
whose derivative is z^{-1}
- The same holds for $\mathbb{C}\backslash(-\infty,0]$, but the two logarithm functions are **not** the same
- The removed curves $[0,\infty)$ and $(-\infty,0]$ are called **branch cuts**