MATH-GA2450 Complex Analysis

Lax's Proof of Cauchy Integral Formula

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October 29, 2024

Cauchy Integral Formula for Analytic Functions

- **Theorem.** If, for each $z \in D(z_0, R)$,

$$f(z) = \sum_{k=0} a_k (z - z_0)^k$$

converges absolutely and $c : [a, b] \to \mathbb{C} \backslash \{z_0\}$ is a closed piecewise C^1 curve, then

$$\frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0}\, dz = W(c, z_0) f(z_0)$$

Cauchy Integral Theorem For Holomorphic Functions

- Let $O \subset \mathbb{C}$ be open and $z_0 \in O$
- Let $c : [a, b] \to O \backslash \{z_0\}$ be a piecewise C^1 closed curve
- Assume that for any $t \in [a, b]$, the line segment from z_0 to $c(t)$ is in O
- This holds if and only if the image of the map

$$C(s, t) = z_0 + s(c(t) - z_0)$$

  is in O
- **Theorem.** If $f : O \to \mathbb{C}$ is holomorphic, then

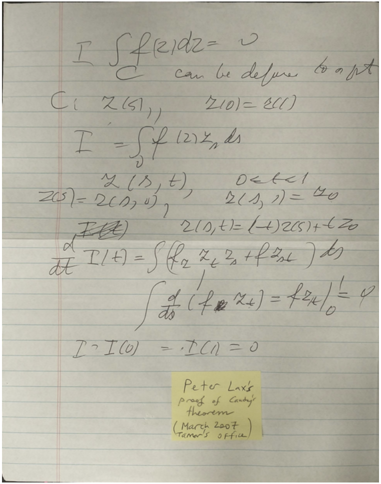
$$\int_c \frac{f(z)}{z - z_0}\, dx = 2\pi W(c, 0) f(0)$$

- It suffices to prove this for $z_0 = 0$

Contour Integral of $C(s,\cdot)$

- For each $s \in [0,1]$, the curve
$$t \mapsto C(s,t) = sc(t)$$
is closed and piecewise C^1
- For each $s \in (0,1]$, let
$$\begin{aligned}
I(s) &= \int_{C(s,\cdot)} \frac{f(z)}{z}\, dz \\
&= \int_{t=a}^{b=t} \frac{f(C(s,t))}{C(s,t)} \partial_t C(s,t)\, dt \\
&= \int_{t=a}^{b=t} \frac{f(sc(t))}{sc(t)} sc'(t)\, dt \\
&= \int_{t=a}^{b=t} \frac{f(sc(t))}{c(t)} c'(t)\, dt
\end{aligned}$$
- Differentiability of $f(sc(t))$, as a function of t, implies
  - $I : (0,1) \to \mathbb{C}$ is differentiable
  - I extends continuously to the domain $[0,1]$

Lax's Proof of the Cauchy Integral Formula



$I = \int_C f(z)\,dz = 0$ C can be deformed to a pt

C: $z(s)$, $z(0) = z(1)$

$I = \int_0^1 f(z) z_s \, ds$

$z(s,t)$, $0 \le t \le 1$

$z(s) = z(s,0)$, $z(s,1) = z_0$

$z(s,t) = (1-t)z(s) + t z_0$

$\frac{d}{dt} I(t) = \int (f_z z_t z_s + f z_{st})\, ds$

$\int_0^1 \frac{d}{ds}(f z_t) = f z_t \Big|_0^1 = 0$

$I = I(0) = I(1) = 0$

Peter Lax's proof of Cauchy's theorem (March 2007 Tamar's office)

Lax's Proof of the Cauchy Integral Formula (Part 1)

- Observe that

$$\begin{aligned} I(0) &= \lim_{s \to 0} \int_{t=a}^{t=b} \frac{f(sc(t))}{c(t)} c'(t)\, dt \\ &= \int_{t=a}^{t=b} \frac{f(0)}{c(t)} c'(t)\, dt \\ &= f(0) \int_c \frac{dz}{z} \\ &= 2\pi i f(0) W(c, 0) \\ I(1) &= \int_{C(1,\cdot)} \frac{f(z)}{z}\, dz \\ &= \int_c \frac{f(z)}{z}\, dz \end{aligned}$$

- It therefore suffices to prove that I is a constant function

Peter Lax's Proof of Cauchy Integral Formula (Part 2)

- If $s \neq 0$, then

$$\begin{aligned} I'(s) &= \frac{d}{ds} \int_{t=a}^{t=b} \frac{f(sc(t))}{c(t)} c'(t)\, dt \\ &= \int_{t=a}^{t=b} \frac{d}{ds} \left(\frac{f(sc(t))}{c(t)} c'(t) \right) dt \\ &= \int_{t=a}^{t=b} \frac{f'(sc(t))c(t)}{c(t)} c'(t)\, dt \\ &= \int_{t=a}^{t=b} f'(sc(t))c'(t)\, dt \\ &= \frac{1}{s} \int_{t=a}^{t=b} \frac{d}{dt} f(sc(t))\, dt \\ &= \frac{1}{s}(f(sc(b)) - f(sc(a))) \\ &= 0 \end{aligned}$$

- This implies that I is constant and therefore $I(0) = I(1)$, proving the formula