MATH-GA2450 Complex Analysis

Cauchy Integral Formula on Convex Domain
Local Cauchy Integral Formula
Holomorphic Implies Analytic
Radius of Convergence for Holomorphic Function

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Cauchy Integral Formula on Convex Domain

- An open $O \subset \mathbb{C}$ is **convex** if for any $z_0, z \in O$, the line segment from z_0 to z is also in O
- **Corollary.** If $f : O \to \mathbb{C}$ is holomorphic, then for any $z \in O$ and piecewise C^1 closed curve c in $O \backslash \{z\}$,

$$\int_c \frac{f(w)}{w - z}\, dw = 2\pi i W(c, z) f(z)$$

Local Cauchy Integral Formula

- Let $O \subset \mathbb{C}$ be open
- Given any $z_0 \in O$, there exists $r > 0$ such that $\overline{D(z_0, r)} \subset O$
- Let $c : [0, 2\pi] \to O \backslash \{z_0\}$ be the parameterization of $\partial D(z_0, r)$ given by
$$c(t) = z_0 + re^{it}$$
- For each $z \in D(z_0, r)$, c is a star-shaped curve around z and therefore
$$W(c, z) = 1$$
- It follows that if $f : O \to \mathbb{C}$ is holomorphic, then
$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(w)}{w - z}\, dw$$

Derivatives of the Cauchy Integral Formula

- Differentiating

$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(w)}{w - z} \, dw$$

with respect to z, we get

$$f'(z) = \frac{1}{2\pi i} \int_c \frac{f(w)}{(w - z)^2} \, dw$$

$$f''(z) = \frac{2(1)}{2\pi i} \int_c \frac{f(w)}{(w - z)^3} \, dw$$

$$f^{(3)}(z) = \frac{3(2)(1)}{2\pi i} \int_c \frac{f(w)}{(w - z)^4} \, dw$$

$$\vdots$$

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_c \frac{f(w)}{(w - z)^{k+1}} \, dw$$

- This implies that f is infinitely differentiable on O and therefore has a Taylor series

Holomorphic Implies Analytic

- **Theorem.** If $O \subset \mathbb{C}$ is open and $f : O \to \mathbb{C}$ is holomorphic, then f is analytic
- Recall that f is analytic if for each $z_0 \in O$, there exists $r > 0$ and a power series
$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$
that converges absolutely on $D(z_0, r)$ and for each $z \in D(z_0, r)$,
$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
- If this holds, then
$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

Key Calculation

- Assume $|w - z_0| = r$ and $|z - z_0| < r$
- Key estimate:

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)}$$
$$= \frac{1}{w - z_0}\left(\frac{1}{1 - \frac{z - z_0}{w - z_0}}\right)$$
$$= \frac{1}{w - z_0}\left(1 + \frac{z - z_0}{w - z_0} + \left(\frac{z - z_0}{w - z_0}\right)^2 + \cdots\right)$$

- This series converges absolutely because

$$\left|\frac{z - z_0}{w - z_0}\right| = \frac{|z - z_0|}{r} < 1$$

Proof that Holomorphic Implies Analytic (Part 1)

- Given $z_0 \in O$, there exists $r > 0$ such that $\overline{D(z_0, r)} \subset O$
- Let $c : [0, 2\pi]$ be a parameterization of $\partial D(z_0, r)$, e.g.,

$$c(t) = z_0 + re^{it}$$

- Given any $z \in D(z_0, r)$, by the local Cauchy integral formula and the key calculation,

$$\begin{aligned}
2\pi i f(z) &= \int_c \frac{f(w)}{w - z}\, dw \\
&= \int_c \frac{f(w)}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k \, dw \\
&= \sum_{k=0}^{\infty} (z - z_0)^k \int_c \frac{f(w)}{(w - z_0)^{k+1}}\, dw \\
&= 2\pi i \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k
\end{aligned}$$

Proof that Holomorphic Implies Analytic (Part 2)

- If $z \in D(z_0, r)$, then for each $w \in \partial D(z_0, r)$, the series
$$\sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k,$$
converges absolutely
- Since $\partial D(z_0, r)$ is compact, it follows that the series converges uniformly with respect to $w \in \partial D(z_0, r)$
- Therefore,
  - The sum and integral can be swapped
  - The resulting series also converges absolutely
- It follows that for any $z \in D(z_0, r)$,
$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$
- A function on an open $O \subset \mathbb{C}$ is holomorphic if and only if it is analytic

Radius of Convergence for an Analytic Function

- The proof also shows that if $D(z_0, r) \subset O$, then the radius of convergence for

$$f(z) = \sum_{k=0}^{k=\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

  is at least r

- Consider an analytic function

$$f(z) = \sum_{k=0}^{k=\infty} a_k(z - z_0)^k$$

- If $R > 0$ is the radius of convergence of f, i.e., the largest R for which the power series converges absolutely on $D(z_0, R)$, then f cannot be extended as a holomorphic function to any disk centered at z_0 with larger radius

Examples

- Let $\log : \mathbb{C} \backslash (-\infty, 0] \to \mathbb{C}$ be the logarithm function with $\log(1) = 0$
- The radius of convergence of the Taylor series for $\log(z)$ centered at $z = 1$ is 1
- The radius of convergence of the Taylor series for $\log(z)$ centered at $z = 2$ is 2
- The radius of convergence of the Taylor series for $\log(z)$ centered at $z = i$ is $\sqrt{2}$