MATH-GA2450 Complex Analysis

Bound on Derivatives of Holomorphic Function
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Bound on Derivatives of a Holomorphic Function

- Let $f : O \to \mathbb{C}$ be holomorphic and $D(z_0, r) \subset O$
- Let $\|f\|_r = \sup\{|f(z)| : \; z \in \partial D(z_0, r)\}$
- If $c(t) = z_0 + re^{it}$, then

$$\begin{aligned}
\left| \frac{f^{(k)}(z_0)}{k!} \right| &= \left| \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^{k+1}} \, dz \right| \\
&= \frac{1}{2\pi} \left| \int_{t=0}^{t=2\pi} \frac{f(c(t))}{r^{k+1} e^{i(k+1)t}} ire^{it} \, dt \right| \\
&\leq \frac{1}{2\pi} \int_{t=0}^{t=2\pi} \frac{|f(c(t))|}{r^k} \, dt \\
&\leq \frac{\|f\|_r}{r^k}
\end{aligned}$$

Liouville's Theorem: A Bounded Entire Function is Constant

- An entire function is a function that is holomorphic on all of $\mathbb{C}$
- Examples
  - Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$
  - $e^{p(z)}$ for any polynomial p
  - $\sin(z)$, $\cos(z)$
- A function f is bounded if there exists $C > 0$ such that $|f(z)| \leq C$ for all z in the domain of f
- **Liouville's Theorem.** Any bounded entire function is a constant function
- **Proof.** For each $z_0 \in \mathbb{C}$,
  - If f is bounded by C, then for any $r > 0$, $\|f\|_r \leq C$
  - Therefore, for any $r > 0$,

    $$|f'(z_0)| \leq \frac{\|f\|_r}{r^k} \leq \frac{C}{r^k},$$

    which implies $f'(z_0) = 0$
  - Since this holds for every $z_0 \in \mathbb{C}$, f is constant

Fundamental Theorem of Algebra

- **Corollary.** Any nonconstant polynomial has at least one complex root
- **Fact.** A nonconstant polynomial is unbounded
  - If $|z| > 1$, then for any $k \geq 1$, $|z|^k > |z|$ and therefore

$$\begin{aligned} |f(z)| &= \left| a_n z^n \left(\frac{b_0}{z^n} + \cdots + \frac{b_{n-1}}{z} + 1 \right) \right| \\ &\leq |a_n||z|^n \left(1 - \frac{|b_0| + \cdots + |b_{n-1}|}{|z|} \right) \end{aligned}$$

  - Therefore, if

$$|z| > R > 2(|b_0| + \cdots + |b_{n-1}|),$$

then

$$|f(z)| > \frac{|a_n|}{2} R^n$$

Proof of Fundamental Theorem of Algebra

- **Proof.** Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ and

$$g(z) = \frac{1}{f(z)}$$

- If $n > 0$, then there exists $R > 0$ such that

$$|z| > R \implies |f(z)| > 1$$

- If f has no roots, there exists $c > 0$ such that if $|z| \leq R$, then

$$|f(z)| > c$$

- Therefore,

$$|g(z)| \leq \max\left(1, \frac{1}{c}\right)$$

- It follows that g is a bounded entire function and therefore constant
- This implies that f is constant

Orientation of a Curve

- A curve c is always a continuous parameterized curve

$$c : [a, b] \to \mathbb{C}$$

- Two nonconstant parameterized curves

$$c_1 : [a_1, b_1] \to \mathbb{C} \text{ and } c_2 : [a_2, b_2] \to \mathbb{C}$$

  parameterize the same curve if there exists a monotone function

$$u : [a_1, b_1] \to [a_2, b_2]$$

  such that $u(a_1) = a_2$, $u(b_1) = b_2$, and, for each $t \in [a_1, b_1]$,

$$c_1(t) = c_2(u(t))$$

- If f is increasing then the two curves have the same orientation
- If f is decreasing then the two curves have opposite orientations

Reverse Orientation of Curve

- Given a curve $c : [a, b] \to \mathbb{C}$, the curve

$$\begin{aligned}(-c) : [b, a] &\to \mathbb{C}\\ t &\mapsto c(t)\end{aligned}$$

  parameterizes the same curve but with the opposite orientation
- If f is holomorphic on an open set containing c, then

$$\begin{aligned}\int_{-c} f(z)\,dz &= \int_{t=b}^{t=a} f(c(t))c'(t)\,dt\\ &= -\int_{t=a}^{t=b} f(c(t))c'(t)\,dt\\ &= -\int_{c} f(z)\,dz\end{aligned}$$

Contour Integral of Oriented Curves

- If $c_1 : [a_1, b_1] \to \mathbb{C}$ and $c_2 : [a_2, b_2] \to \mathbb{C}$ parameterize the same curve and have the same orientation, then for any holomorphic f,

$$\begin{aligned}\int_{c_2} f(z)\,dz &= \int_{u=a_2}^{u=b_2} f(c_2(u))c_2'(u)\,du \\ &= \int_{t=a_1}^{t=b_1} f(c_2(u(t)))c_2'(u(t))u'(t)\,dt \\ &= \int_{t=a_1}^{t=b_1} f(c_1(t))c_1'(t)\,dt \\ &= \int_{c_1} f(z)\,dz\end{aligned}$$

- A similar calculation shows that if c_1 and c_2 have opposite orientations, then

$$\int_{c_2} f(z)\,dz = -\int_{c_1} f(z)\,dz$$