MATH-GA2450 Complex Analysis

Analytic Isomorphisms

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Analytic Isomorphisms

- **Theorem.** If $U \subset \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is holomorphic and injective, then for every $z \in U$,

$$f'(z) \neq 0$$

and the inverse map

$$f^{-1} : f(U) \to U$$

is also holomorphic
- Such a map is called an **analytic isomorphism**
- If $U, V \subset \mathbb{C}$ are open, then they are **analytically isomorphic** is there exists an analytic isomorphism

$$f : U \to V$$

such that $f(U) = V$
- An analytic isomorphism $f : U \to U$ is an **analytic automorphism**
- Let $\mathrm{Aut}(U)$ denote the space of all analytic isomorphisms

Sketch of Proof

- For each $z_0 \in U$, f is analytic and therefore has a power series

$$f(z) = \sum_{k=n} a_k (z - z_0)^k$$

- If $m > 1$, then

$$f(z) = a_n(z-z_0)^n(1+b_1(z-z_0)+b_2(z-z_0)^2+\cdots) \simeq a_n(z-z_0)^n,$$

which is not injective

Basic Properties of Analytic Isomorphisms

- If $f : U \to V$ and $g : V \to W$ are isomorphisms, then so is

$$g \circ f : U \to W$$

- If $f : U \to V$ is an isomorphisim, then so is

$$f^{-1} : V \to U$$

- If $f, g : U \to V$ are isomorphisms, then there exists $h \in \mathrm{Aut}(V)$ such that

$$g = h \circ f$$

- If U, V are isomorphic, there is a bijection

$$\mathrm{Aut}(U) \rightleftarrows \mathrm{Aut}(V)$$

- In particular, if $f : U \to V$ is an isomorphism and $g : U \to U$ is a map, then

$$g \in \mathrm{Aut}(U) \iff f \circ g \circ f^{-1} \in \mathrm{Aut}(V)$$

Aut(U) is a Group

- **Group multiplication:** $f, g \in \text{Aut}(U) \implies f \circ g \in \text{Aut}(U)$
- **Associativity:** If $f, g, h \in \text{Aut}(U)$, then

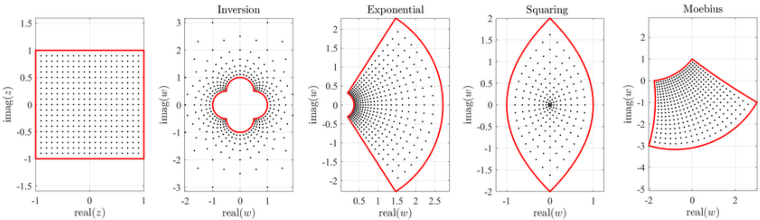
$$(f \circ g) \circ h = f \circ (g \circ h)$$

- **Identity element:** The map $I : U \to U$ given by

$$I(z) = z$$

  is an isomorphism such that for any $f \in \text{Aut}(U)$, $f \circ I = I \circ f = f$
- **Inverse element:** For any $f \in \text{Aut}(U)$, $f^{-1} \in \text{Aut}(U)$

Riemann Mapping Theorem



- Let $D = D(0,1)$
- Let $U \subsetneq \mathbb{C}$ be open
- **Theorem.** There exists an analytic isomorphism

$$f : U \to D$$

- **Corollary.** If $U, V \subsetneq \mathbb{C}$ are open, then they are analytically isomorphic

Upper Half-Plane is Isomorphic to Disk (Part 1)

- The upper half-plane is

$$H = \{x + iy \in \mathbb{C} : \ y > 0\}$$

- **Theorem.** The map

$$f(z) = \frac{z - i}{z + i}$$

  is an analytic isomorphism from H to D
- Observe that

$$f(x + iy) = \frac{x + i(y - 1)}{x + i(y + 1)}$$

- If $y > 0$, then $(y - 1)^2 < (y + 1)^2$ and therefore

$$|f(x + iy)|^2 = \frac{|x + i(y - 1)|^2}{x + i(y + 1)|^2} = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2} < 1$$

- Therefore, $f(H) \subset D$

Upper Half-Plane is Isomorphic to Disk (Part 2)

- If
$$w = \frac{z - i}{z + i},$$
then
$$wz + iw = z - i$$
and therefore
$$z = i\frac{1 + w}{1 - w} = i\frac{(1 + w)(1 - \bar{w})}{|1 - w|^2} = i\frac{1 - |w|^2 + w - \bar{w}}{|1 - w|^2}$$
- If $w \in D$, then $1 - |w|^2 > 0$ and therefore the imaginary part of z is
$$\text{im}(z) = \frac{1 - |w|^2}{|1 - w|^2} > 0$$
- It follows that $f^{-1}(D) \subset H$
- This implies that $f(H) = D$ and $f^{-1}(D) = H$

Analytic Isomorphism from First Quadrant to Disk

- Let
$$Q = \{x + iy : \; x, y > 0\}$$
- Observe that the map $g(z) = z^2$ is an analytic isomorphism from Q to H
- Therefore, if $f : H \to D$ is the analytic isomorphism from above, then the map
$$f \circ g(z) = f(z^2) = \frac{z^2 - i}{z^2 + i}$$
is an analytic isomorphism from Q to D

Automorphisms of Disk: Rotations

- A basic question is what are the analytic automorphisms of the unit disk?
- Given $\phi \in \mathbb{R}$, the function $R_\phi : D \to D$ given by

$$R(z) = e^{i\phi} z$$

  is an analytic isomorphism of D that rotates each z counterclockwise by angle ϕ

Automorphisms of Disk: Rescale Upper Half Plane

- Given any $\rho \in (0, \infty)$, the function $S_\rho : H \to H$ given by

$$S_\rho(z) = \rho z$$

is an isomorphism of H that rescales each z by a factor of ρ

- This defines an isomorphism of D given by

$$\begin{aligned} f \circ S_\rho \circ f^{-1}(z) &= f \circ S_\rho \left(i \frac{1+z}{1-z} \right) = f \left(i\rho \frac{1+z}{1-z} \right) \\ &= \frac{i\rho \frac{1+z}{1-z} - i}{i\rho \frac{1+z}{1-z} + i} = \frac{\rho(1+z) - (1-z)}{\rho(1+z) + 1 - z} \\ &= \frac{(\rho+1)z + \rho - 1}{(\rho-1)z + \rho + 1} = \frac{z + \alpha}{1 + \alpha z}, \end{aligned}$$

where $\alpha \in (0, 1)$

Automorphisms of Disk: Shift Upper Half Plane

- Given any $t \in \mathbb{R}$, the function $T_t : H \to H$ given by

$$T_t(z) = z - t$$

  is an isomorphism of H that shifts each z horizontally by t
- This defines an isomorphism of D given by

$$\begin{aligned}
f \circ T_t \circ f^{-1}(z) &= f \circ T_t \left(i \frac{1+z}{1-z} \right) = f \left(i \frac{1+z}{1-z} - t \right) \\
&= f \left(\frac{i(1+z) - t(1-z)}{1-z} \right) = \frac{\frac{(i-t)z+i-t}{1-z} - i}{\frac{(i-t)z+i-t}{1-z} + i} \\
&= \frac{(i-t)z + i - t - i(1-z)}{(i-t)z + i - t + i(1-z)} = \frac{(2i-t)z - t}{-tz + 2i - t} \\
&= \frac{2i-t}{2i-t} \left(\frac{z-\alpha}{1-\bar{\alpha}z} \right), \ \alpha = \frac{t}{2i-t}
\end{aligned}$$

Analytic Function Not Injective at Critical Point (Part 1)

- Recall that if $f(z) = (z - z_0)^n$, then for any $r > 0$ and $0 \leq k \leq n - 1$,

  $$z_1 = e^{\frac{i2\pi}{n}}, \ldots, z_{n-1} = e^{\frac{2\pi(n-1)}{n}}$$

  are n distinct values such that

  $$f(z_0 + re^{\frac{i2\pi}{n}}) = r^n$$

  and therefore if $n \geq 2$, f is not injective for any $D(z_0, r)$

Analytic Function Not Injective at Critical Point (Part 2)

- Let $O \subset \mathbb{C}$ be open and $f : O \to \mathbb{C}$ be holomorphic
- **Theorem.** If $z_0 \in O$ is a critical point of f, then for any $r > 0$, $f : D(z_0, r) \to \mathbb{C}$ is not injective

Proof (Part 1)

- For simplicity, assume that $f(z_0) = a_0 = 0$
- Since $f'(z_0) = 0$,

$$f(z) = \sum_{k=2}^{\infty} a_k (z - z_0)^k$$

- If, for every $k \geq 2$, $a_k = 0$, then f is constant and therefore not injective
- Can therefore ssume there exists $n \geq 2$ such that $a_n \neq 0$ and

$$\begin{aligned} f(z) &= \sum_{k=n}^{\infty} a_k (z - z_0)^k \\ &= a_n (z - z_0)^n \left(1 + \sum_{k=1} b_{n+k} (z - z_0)^k \right), \end{aligned}$$

where $b_{n+k} = \frac{a_{n+k}}{a_n}$

Proof (Part 2)

- There exists $R_0 > 0$ be such that $\overline{D(z_0, R_0)} \subset O$ and for all $z \in D(z_0, R_0)$,

$$\left| \sum_{k=1} b_{n+k}(z - z_0)^k \right| < \frac{1}{2}$$

- Therefore, for any $z \in \partial D(z_0, R_0)$

$$\frac{1}{2}|a_n||z - z_0| \le |f(z)| \le |a_n||z - z_0|^n$$

Proof (Part 3)

- Since $f'(z_0) = 0$ and is analytic, it has a power series

$$f'(z) = (z - z_0) \sum_{k=0} c_k (z - z_0)^k$$

and therefore there exists $c' \geq 0$ such that

$$|f'(z)| \leq c' |z - z_0|^k$$

Proof (Part 4)

- For any $0 < r < R$, there exists $z_1 \in D(z_0, r)$ such that

$$f'(z_1) \neq 0$$

Otherwise, f is constant on $D(z_0, r)$ and therefore not injective

- On the other hand, since for any $z \in D(z_0, R)$,

$$|f(z)| > \frac{1}{2}|a_n||z - z_0|^n,$$

the only zero of f in $D(z_0, R)$ is z_0 and

$$n = \int_{\partial D(z_0,R)} \frac{f'(z)}{f(z) - f(z_0)}\, dz$$

Proof (Part 5)

- If $g(z) = f(z) - f(z_1)$, then

$$f(z) = f(z_1) \iff g(z) = 0$$

- If N is the number of zeros in $D(z_0, R)$ of $g(z)$, then

$$2\pi i N = \int_{\partial D(z_0),R)} \frac{g'(z)}{g(z)}\, dz = \int_{\partial D(z_0),R)} \frac{f'(z)}{f(z) - f(z_1)}\, dz$$

- Observe that since $|z_1 - z_0| = r$, if $z \in \partial D(z_0, R)$ and

$$r < \frac{1}{4^{1/n}} R,$$

$$|f(z) - f(z_1)| \geq |f(z)| - |f(z_1)| \geq |a_n| \left(\frac{1}{2} R^n - r^n \right) \geq \frac{1}{4} R^n$$

Proof (Part 6)

- Therefore,

$$\begin{aligned}|2\pi i(N-n)| &= \left|\int_{\partial D(z_0,R)} \frac{f'(z)}{f(z)-f(z_1)} - \frac{f'(z)}{f(z)}\,dz\right| \\ &\leq \int_{\partial D(z_0,R)} \left|\frac{f'(z)}{f(z)}\right| \left|\frac{f(z_1)}{f(z)-f(z_1)}\right| dz \\ &\leq 2\pi r \frac{c' R^n}{\frac{1}{2}|a_n| R^n} \frac{|a_n| r^n}{\frac{1}{4}|a_n| R^n} \\ &= \frac{8\pi r^{n+1}}{R^n}\end{aligned}$$

- Since this holds for any $r < R$, follows that $N = n$
- Since $N = n \geq 1$ and the order of the zero at z_1 is 1, the number of distinct zeros of g has to be at least 2
- It follows that f is not injective on $D(z_0, r)$ for any $r > 0$ such that $D(z_0, r) \subset O$