MATH-GA2450 Complex Analysis

Geometry of Sphere
Geometry of Hyperbolic Plane

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Geometry of the Unit Sphere in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$

- The standard unit sphere is

$$S^2 = \{(x, y, z) : \ x^2 + y^2 + z^1 = 1\}$$
$$= \{(\zeta, h) : \ |\zeta|^2 + h^2 = 1\}$$

- A **great circle** is the intersection of the sphere with a plane that contains the origin
- Two points p, q are **antipodal** if $q = -p$
- If two points are not antipodal, there is a unique plane that contains the two points and the origin
- Therefore, there is a unique great circle that contains p and q
- The shortest path between any two points $p, q \in S^2$ is the shorter arc connecting p to q of the great circle containing them
- The **spherical distance** between p and q is the length of the arc

Rotations of the Sphere

- A map $R : S^2 \to S^2$ is an **isometry** if for any $p, q \in S^2$, if the distance between $R(p)$ and $R(q)$ is equal to the distance between p and q
- Given a line through the origin and $\theta \in \mathbb{R}$, the sphere can be rotated around the line by angle θ
- Such a map is called a **rotation**
- Any isometry of S^2 is a rotation
- An isometry is a conformal map

Stereographic Projection of Sphere

- Let
$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$
denote the Riemann sphere
- The standard unit sphere in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ is
$$S^2 = \{(\zeta, h) \in \mathbb{C} \times \mathbb{R} : \ |\zeta|^2 + h^2 = 1\}.$$
- The north pole is $N = (0, 1)$
- Stereographic projection projects a point on the unit sphere to a point on the Riemann sphere
- The stereographic projection of $p = (\zeta, h) \in S^2$, where $h \neq 1$ is the intersection of the line that contains N and p with the complex plane

Formulas for Stereographic Projection and Its Inverse

- The formula for stereographic projection is

$$\pi : S^2 \to \widehat{\mathbb{C}}$$
$$(\zeta, h) \mapsto \begin{cases} \dfrac{\zeta}{1-h} & \text{if } h \neq 1 \\ \infty & \text{if } h = 1 \end{cases}.$$

- The inverse map is

$$\pi^{-1} : \widehat{\mathbb{C}} \to S^2$$
$$z \mapsto \begin{cases} \left(\dfrac{2z}{1+|z|^2}, \dfrac{-1+|z|^2}{1+|z|^2} \right) & \text{if } z \neq \infty \\ (0,1) & \text{if } z = \infty \end{cases}.$$

- Stereographic projection and its inverse are conformal maps

Stereographic Projection of Antipodal Points

- Two points $p, q \in S^2$ are **antipodal** if $q = -p$
- The intersection of any line through the origin with S^2 consists of antipodal points
- Observe that

$$\pi^{-1}(0) = (0, -1) \text{ and } \pi^{-1}(\infty) = (0, 1)$$

  are antipodal points
- If $z \neq 0, \infty$ and $p = \pi^{-1}(z)$, observe that

$$\begin{aligned}
\pi^{-1}(-\bar{z}^{-1}) &= \left(\frac{-2\bar{z}^{-1}}{1 + |z|^{-2}}, \frac{-1 + |z|^{-2}}{1 + |z|^{-2}} \right) \\
&= \left(\frac{-2z}{1 + |z|^2}, \frac{-|z|^2 + 1}{|z|^2 + 1} \right) \\
&= -\pi^{-1}(z) \\
&= -p
\end{aligned}$$

- In other words if $z = \pi(p)$, then $\bar{z}^{-1} = \pi(-p)$

Great Circles Containing North and South Poles

- It is easy to check that if C is a great circle obtained by intersecting S^2 with a vertical plane, then $\pi(C)$ is a line through the origin

Stereographic Projection of a Rotation

- Any rotation $R : S^2 \to S^2$ by angle θ defines a corresponding map

$$\begin{aligned} \widehat{R} : \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ z &\mapsto \pi \circ R \circ \pi^{-1}(z) \end{aligned}$$

- **Goal:** Find the formula for $\widehat{R}$

Rotation Around the Vertical Axis

- A rotation R_θ of the sphere around the axis through the north and south poles by angle θ is given by

$$R_\theta(\zeta, h) \mapsto (e^{i\theta}\zeta, h)$$

- It is easy to check that the corresponding automorphism of $\widehat{\mathbb{C}}$ is given by

$$\widehat{R}_\theta(z) = e^{i\theta} z$$

Rotation Around Axis Through $\{p, -p\}$ by angle θ

- Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a conformal map that maps the axis through $\{0, 1, \infty\}$ to the axis through $\{p, 0, -p\}$
- To rotate by angle θ around the axis through $\{p, 0, -p\}$,
    - Use F^{-1} to map axis through $\{p, 0, -p\}$ to axis through $\{0, 1, \infty\}$
    - Use $\widehat{R}_\theta$ to rotate by angle θ around axis through $\{0, 1, \infty\}$
    - Use F to map axis $\{0, 1, \infty\}$ back to $\{p, 0, -p\}$

Conformal Map from $\{0, 1, \infty\}$ to $\{p, 0, -p\}$ (Part 1)

- Given $p \in S^2$, let $z_0 = \pi(p)$ and therefore $-\bar{z}_0^{-1} = \pi(-p)$
- Consider a conformal map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$F(0) = z_0$$
$$F(\infty) = -\bar{z}_0^{-1}$$
$$F(1) = 0$$

- If $\ell \subset \widehat{\mathbb{C}}$ is the line through $0, 1, \infty$, then $\pi^{-1}(\ell)$ is the great circle passing through

$$\pi^{-1}(0) = (0, -1), \pi^{-1}(1) = (1, 0), \pi^{-1}(\infty) = (0, 1)$$

- $F(\ell)$ is the line passing through $z_0, 0, \bar{z}_0^{-1}$
- $\pi^{-1}(F(\ell))$ is the great circle passing through

$$\pi^{-1}(z_0) = p, \pi^{-1}(0) = (0, -1), \pi^{-1}(-\bar{z}_0^{-1}) = -p$$

Conformal Map from $\{0, 1, \infty\}$ to $\{p, 0, -p\}$ (Part 2)

- The conformal map

$$F(z) = \frac{az + b}{cz + d}$$

  satisfies

$$F(0) = \frac{b}{d} = z_0$$
$$F(\infty) = \frac{a}{c} = -\bar{z}_0^{-1}$$
$$F(1) = \frac{a + b}{c + d} = 0$$

- If $a = 1$, then $b = -1$ and

$$c = -\bar{z}_0$$
$$d = -z_0^{-1}$$

Conformal Map from $\{0, 1, \infty\}$ to $\{p, 0, -p\}$ (Part 3)

- Therefore,

$$F(z) = \frac{z-1}{-z_0^{-1} - \bar{z}_0 z} = \frac{1-z}{z_0^{-1} + \bar{z}_0 z} = \frac{z_0(1-z)}{1+|z_0|^2 z}$$

- If

$$w = F(z) = \frac{z_0(1-z)}{1+|z_0|^2 z}$$

then

$$z = \frac{1 - z_0^{-1} w}{1 + w\bar{z}_0)}$$

- Therefore,

$$F^{-1}(z) = \frac{1 - z_0^{-1} z}{1 + z\bar{z}_0}$$

Rotation Around Axis Through $\{p, -p\}$ by angle θ

- Therefore, rotation by angle θ around the axis through $\{p, 0, -p\}$ is given by

$$
\begin{aligned}
F \circ R_\theta \circ F^{-1}(z) &= F\left(e^{i\theta}\frac{1 - z_0^{-1}z}{1 + z\bar{z}_0}\right) \\
&= \frac{z_0\left(1 - e^{i\theta}\frac{1 - z_0^{-1}z}{1 + z\bar{z}_0}\right)}{1 + |z_0|^2 e^{i\theta}\frac{1 - z_0^{-1}z}{1 + z\bar{z}_0}} \\
&= \frac{z_0(1 + z\bar{z}_0 - e^{i\theta}(1 - z_0^{-1}z))}{1 + z\bar{z}_0 + |z_0|^2 e^{i\theta}(1 - z_0^{-1}z)} \\
&= \frac{z_0(1 + z\bar{z}_0) + e^{i\theta}(z - z_0)}{1 + z\bar{z}_0 - \bar{z}_0 e^{i\theta}(z - z_0)}
\end{aligned}
$$

Sphere Via Dot Product

- Recall that the dot product of $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ is defined to be
$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$
- The distance $d(p_1, p_2)$ between two points $p_1, p_2 \in \mathbb{R}^3$ is, by the Pythagorean Theorem, the square root of
$$(d(p_1, p_2))^2 = (p_1 - p_2) \cdot (p_1 - p_2)$$
- The unit sphere is defined to be
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : \ (x, y, z) \cdot (x, y, z) = 1\}$$
- Spherical coordinates:
$$(x, y, z) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$
- Stereographic coordinates: Given $\xi \in \widehat{\mathbb{C}}$,
$$(x, y, z) = \left(\frac{2\xi}{1 + |\xi|^2}, \frac{|\xi|^2 + 1}{|\xi|^2 + 1} \right)$$

Minkowski Space

- The **Minkowski product** of $p_1 = (x_1, y_1, t_1)$ and $p_2 = (x_2, y_2, t_2)$ is defined to be

$$\langle p_1, p_2 \rangle = -t_1 t_2 + x_1 x_2 + y_1 y_2$$

- The **spacetime distance** between p_1, p_2, $d(p_1, p_2)$ is the square root of

$$(d(p_1, p_2))^2 = \langle p_1 - p_2, p_1 - p_2 \rangle = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$$

- $\mathbb{R}^3$ with the Minkowski product is called $(2, 1)$-dimensional Minkowski space
- $\mathbb{R}^4$ with the Minkowski product is called $(3, 1)$-dimensional Minkowski space
  - This is the space-time used by Einstein in his special theory of relativity
  - Here, the units are chosen so that the speed of light is equal to 1

Hyperbolic Sphere With Imaginary Radius

- Given points p, q in space-time, if

$$\langle p, q \rangle = -\tau^2,$$

  then τ is the time required to travel from p to q at the speed of light
- The hyperbolic sphere of radius i is defined to be

$$\mathcal{H} = \{p = (x, y, t) : \langle p, p \rangle = -t^2 + x^2 + y^2 = -1 \text{ and } t > 0\}$$

- Since $t^2 = x^2 + y^2 + 1$, if $(x, y, t) \in \mathcal{H}$, $t \geq 1$
- $\mathcal{H}$ is the upper half of the 2-sheeted hyperboloid
- It is the space of all points in Minkowski space that can be reached by traveling forward in time at the speed of light for 1 unit of time
- $\mathcal{H}$ is the relativistic analogue of the unit sphere in Newtonian mechanics

Geodesics and Isometries

- A geodesic in $\mathcal{H}$ is the intersection of $\mathcal{H}$ with a plane that contains the origin
- Given any two different points $p, q \in \mathcal{H}$, there exists a unique plane containing $0, p, q$ and therefore a unique geodesic passing through p and q
- The length of the geodesic arc from p to q is the **hyperbolic distance from p to q**, denoted $d_{\mathcal{H}}(p, q)$
- A map $\Phi : \mathcal{H} \to \mathcal{H}$ is an **isometry** if for any $p, q \in \mathcal{H}$,

$$d_{\mathcal{H}}(\Phi(p), \Phi(q)) = d_{\mathcal{H}}(p, q)$$

Stereographic projection

- Stereographic projection is defined to be the map
$$\pi : \mathcal{H} \to \mathbb{C},$$
where for each $(x, y, t) \in \mathcal{H}$,
$$\pi(x, y, t) = z \in \mathbb{C},$$
where if $z = u + iv$, then the line through (x, y, t) and $(0, 0, 0)$ intersects the plane $t = 1$ at $(u, v, 1)$
- The formula is
$$\pi(x, y, t) = \frac{x + iy}{1 + t}$$
and
$$\pi^{-1}(z) = \left(\frac{2z}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right)$$
- Observe that
$$|\pi(x, y, t)|^2 = \frac{x^2 + y^2}{(1 + t)^2} = \frac{t^2 - 1}{(t + 1)^2} = \frac{t - 1}{t + 1} < 1$$
- It follows that $\pi(\mathcal{H}) = D$

Hyperbolic Isometries

- A map $\Phi : \mathcal{H} \to \mathcal{H}$ is an isometry if and only if there exists a conformal automorphism
$$f : D \to D$$
such that
$$\Phi = \pi^{-1} \circ f \circ \pi$$
- Recall that the map F given by
$$F(z) = \frac{z - i}{z + i}$$
is an isomorphism from the upper half-plane H to the unit disk D
- Therefore, a map $\Phi : \mathcal{H} \to \mathcal{H}$ is an isometry if and only if there exists an conformal automorphism
$$f : H \to H$$
such that
$$\Phi = \pi^{-1} \circ F \circ f \circ F^{-1} \circ \pi$$

Basic Hyperbolic Isometries

- Rotation by angle θ around t-axis
  - Rotation of disk D around 0
- **Boost:** Hyperbolic rotation by a hyperbolic angle τ
  - Composition of horizontal translation with scaling by a positive real factor